A Frame Decomposition of the Funk-Radon Transform

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Funk–Radon Transform

- Sphere $\mathbb{S}^2 = \{\boldsymbol{\xi} \in \mathbb{R}^3 : \|\boldsymbol{\xi}\| = 1\}$
- Function $f\colon \mathbb{S}^2 \to \mathbb{C}$
- Funk–Radon transform (or spherical Radon transform)

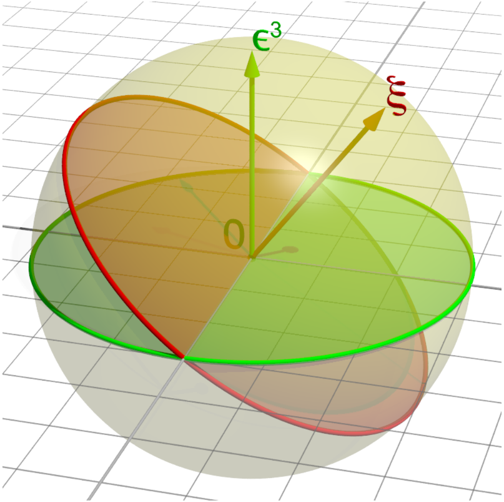
$$\mathcal{R}f(\boldsymbol{\xi}) := \frac{1}{2\pi} \int_{\boldsymbol{\xi}^\top \boldsymbol{\eta}=0} f(\boldsymbol{\eta}) \mathrm{d}\boldsymbol{\eta}, \qquad \forall \boldsymbol{\xi} \in \mathbb{S}^2.$$

(integrals of f along all great circles)

Goal

Reconstruct the function f from integrals $\mathcal{F}f$

- Possible for even functions $f(\boldsymbol{\xi}) = f(-\boldsymbol{\xi})$



Funk–Radon Transform

- Applications
- Diffusion MRI [Tuch 2004] [Rauff et al. 2022]
- Geometric tomography [Gardner 2006]
- Synthetic aperture radar [Yarman Yazici 2011]
- Photoacoustic tomography [Hristova Moon Steinhauer 2016]
- Comption camera imaging [Terzioglu 2023]

- Inversion methods
- Singular value decomposition via spherical harmonics [Minkowski 1904] [Funk 1911]
- Analytic inversion formulas [Funk 1913] [Helgason 1990] [Gindikin Reeds Shepp 1994] [Bailey et al. 2003] [Salman 2016] [Kazantsev 2018]
- Mollifier methods [Louis et al. 2011] [Riplinger, Spiess 2013]
- Discretization on the cubed sphere [Bellet 2023]

Singular Value Decomposition (SVD)

The **singular value decomposition** of a bounded linear operator $A\colon X \to Y$ between Hilbert spaces X and Y has the form

$$Ax = \sum_{k=1}^{\infty} \sigma_k \left\langle x, u_k \right\rangle_X v_k\,, \qquad \forall\, x \in X\,,$$

where $\{u_k\}_{k\in\mathbb{N}}$ and $\{v_k\}_{k\in\mathbb{N}}$ are orthonormal bases over X and Y.

Inversion:

$$A^{\dagger} y = \sum_{k=1}^{\infty} \frac{1}{\sigma_k} \left\langle y, v_k \right\rangle_Y u_k$$

SVD of the Funk-Radon transform [Minkowski 1904]

Denoting by Y_ℓ^m the **spherical harmonics** of degree $\ell \in \mathbb{N}_0$ and order $m = -\ell, \dots, \ell$, we have

$$\mathcal{R} Y_\ell^m = \begin{cases} \frac{(-1)^{\ell/2}(\ell-1)!!}{\ell!!}\, Y_\ell^m\,, & \ell \text{ even}\,,\\ 0\,, & \ell \text{ odd}\,. \end{cases}$$

Frame Decompositions

A **frame decomposition** of a bounded linear operator $A\colon X \to Y$ between Hilbert spaces X and Y has the form

$$Ax = \sum_{k=1}^{\infty} \sigma_k \left\langle x, e_k \right\rangle_X \tilde{f}_k \,, \qquad \forall\, x \in X \,,$$

where $\{e_k\}_{k\in\mathbb{N}}$ and $\{f_k\}_{k\in\mathbb{N}}$ form frames over X and Y, and $\{\tilde{f}_k\}_{k\in\mathbb{N}}$ is the dual frame of $\{f_k\}_{k\in\mathbb{N}}$.
The main requirement on e_k and f_k is the quasi-singular relation

$$\overline{\sigma_k}\, e_k = A^* f_k \,, \qquad \forall\, k \in \mathbb{N} \,. \tag{1}$$

Advantage: More flexible than an SVD (singular value decomposition), but retains important properties: approximate solutions of $Ax = y$, filter-based regularization [Frikel Haltmeier 2020] [Hubmer Ramlau 2021] [Hubmer Ramlau Weissinger 2022] [Ebner et al. 2023]

Question: Can frames satisfying (1) be found?
Possible in many examples [Donoho 1995] [Frikel 2013] [Frikel Haltmeier 2018] [Hubmer Ramlau 2020]

Background: Frames in Hilbert Spaces

[Christensen 2016] [Daubechies 1992]

A sequence $\{e_k\}_{k\in\mathbb{N}}$ in a Hilbert space X is called a **frame** if there exist frame bounds $B_1, B_2 > 0$ such that

$$B_1 \, \|x\|_X^2 \leq \sum_{k=1}^{\infty} \left| \langle x, e_k \rangle_X \right|^2 \leq B_2 \, \|x\|_X^2 \,, \qquad \forall\, x \in X\,.$$

Furthermore, we define

$$Sx := \sum_{k=1}^{\infty} \langle x, e_k \rangle_X \, e_k \,,$$

and the **dual frame** $\tilde{e}_k := S^{-1} e_k$, which forms a frame over X with frame bounds B_2^{-1}, B_1^{-1}. Then

$$x = \sum_{k=1}^{\infty} \langle x, \tilde{e}_k \rangle_X \, e_k = \sum_{k=1}^{\infty} \langle x, e_k \rangle_X \, \tilde{e}_k \,, \qquad \forall\, x \in X\,.$$

Construction of Frame Decompositions

Need two conditions on the bounded linear operator $A\colon X \to Y$ between Hilbert spaces X, Y:

1. For some $c_1, c_2 > 0$ and a Hilbert space $Z \subseteq Y$:

$$c_1 \|x\|_X \leq \|Ax\|_Z \leq c_2 \|x\|_X , \qquad \forall x \in X ,$$

2. Z is a dense subspace of Y, and $\{f_k\}$ is a frame on Y with $f_k \in Z$ (i.e. $\|f_k\|_Z < \infty$) and

$$\|y\|_Y \leq \|y\|_Z , \qquad \forall y \in Z .$$

Theorem [Hubmer Ramlau 2021]

Under the above assumptions, the unique solution of $Ax = y$ is

$$A^{\ddagger} y := \sum_{k=1}^{\infty} \langle Ly, f_k \rangle_Y \tilde{e}_k ,$$

where

$$e_k := A^* L f_k , \qquad L := (EE^*)^{-1/2} ,$$

and $E\colon Z \to Y$ denotes the embedding operator.

Select Suitable Spaces

The **Sobolev space** $H^s(\mathbb{S}^2)$, $s \in \mathbb{R}$, is the completion of $C^\infty(\mathbb{S}^2)$ with respect to the norm

$$\|f\|_{H^s(\mathbb{S}^2)} := \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} (\ell + \tfrac{1}{2})^{2s} |\langle f, Y_\ell^m \rangle_{L^2(\mathbb{S}^2)}|^2,$$

where $\langle f, g\rangle_{L^2(\mathbb{S}^2)} := \int_{\mathbb{S}^2} f(\boldsymbol{\xi}) g(\boldsymbol{\xi}) \, \mathrm{d}\boldsymbol{\xi}$. Denote by $H^s_{\text{even}}(\mathbb{S}^2)$ its restriction to even functions.

Lemma — first part by [Strichartz 1982]

The Funk-Radon transform is a continuous, bijective operator

$$R \colon X = L^2_{\text{even}}(\mathbb{S}^2) \to H^{1/2}_{\text{even}}(\mathbb{S}^2) = Z.$$

It satisfies condition ❶ with the bounds $c_1 = \sqrt{1/2}$ and $c_2 = \sqrt{2/\pi}$.

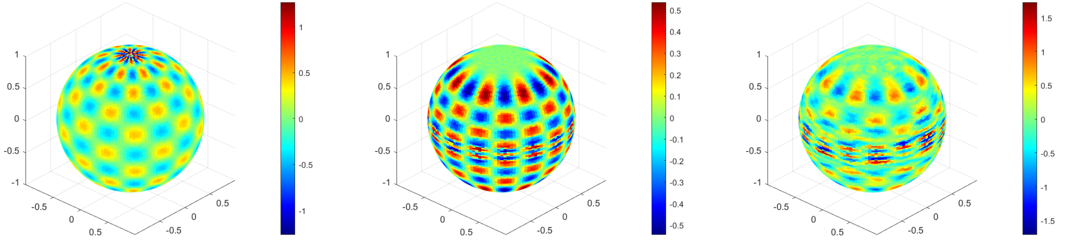
Searching for a Suitable Frame

Define the trigonometric basis

$$b_{n,k} : \mathbb{S}^2 \to \mathbb{R}\,, \; b_{n,k}(\varphi,\theta) := \frac{e^{in\varphi}\,\sin(k\theta)}{\pi\sqrt{\sin\theta}}\,, \qquad \forall\, n \in \mathbb{Z}\,,\; k \in \mathbb{N}\,.$$

Lemma [Q., Weissinger, Hubmer, Erchinger 2023]

The sequence $\{b_{n,k}\}_{n\in\mathbb{Z},k\in\mathbb{N}}$ forms an orthonormal basis of $Y = L^2(\mathbb{S}^2)$ and $b_{n,k} \in H^{1/2}(\mathbb{S}^2)$.



(a) Frame $b_{n,k}$ (b) Frame $e_{n,k} = RLb_{n,k}$ (c) Dual frame $\tilde{e}_{n,k}$

Frame Inversion of the Funk-Radon Transform

Theorem [Q., Weissinger, Hubmer, Erchinger 2023]

Let $E\colon H^{1/2}_{\text{even}}(\mathbb{S}^2) \to L^2_{\text{even}}(\mathbb{S}^2)$ be the embedding operator, and $L \coloneqq (EE^*)^{-1/2}$. Then

$$e_{n,k} \coloneqq RLb_{n,k}\,, \qquad (n,k) \in J \coloneqq \{(n,k) \in \mathbb{Z} \times \mathbb{N} : n+k \text{ odd}\}\,,$$

is a frame in $L^2_{\text{even}}(\mathbb{S}^2)$.
For any $g \in H^{1/2}_{\text{even}}(\mathbb{S}^2)$, the unique solution $f \in L^2_{\text{even}}(\mathbb{S}^2)$ of the inverse problem $Rf = g$ satisfies

$$f = R^{\ddagger} g \coloneqq \sum_{(n,k)\in J} \langle Lg, b_{n,k}\rangle_{L^2(\mathbb{S}^2)}\, \tilde{e}_{n,k}\,,$$

where $\tilde{e}_{n,k}$ is the dual frame of $e_{n,k}$. It holds that $\|R^{\ddagger} g\|_{L^2(\mathbb{S}^2)} \leq 2\,\|Lg\|_{L^2(\mathbb{S}^2)}$.

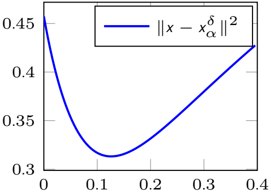
Filter-Based Regularization

With $\lambda_\ell = (\ell + \frac{1}{2})^{-1/2}$, we have

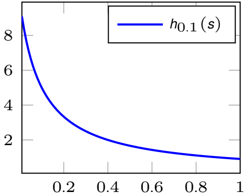
$$Ly = (EE^*)^{-1/2} y = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{1}{\lambda_\ell} \langle y, Y_\ell^m \rangle_{L^2(\mathbb{S}^2)} Y_\ell^m .$$

For noisy data y^δ, define the regularized solution

$$x_\alpha^\delta := R^\ddagger U_\alpha y^\delta, \quad \text{and} \quad LU_\alpha y^\delta := \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \lambda_\ell h_\alpha(\lambda_\ell^2) \langle y^\delta, Y_\ell^m \rangle_{L^2(\mathbb{S}^2)} Y_\ell^m .$$

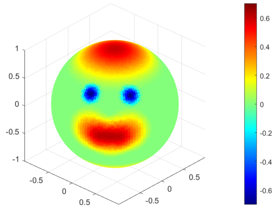


L^2 Error depending on regularization parameter α

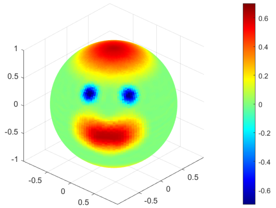


Filter function $h_\alpha(s) = 1/(\alpha + s)$

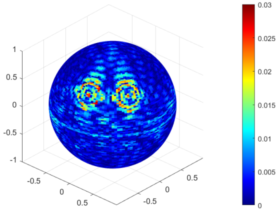
Reconstruction Evaluation, Exact Data



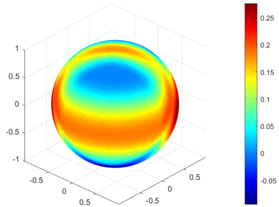
(a) The test function x



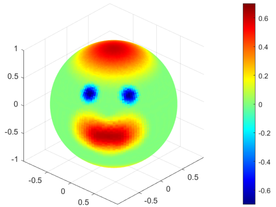
(b) $R^\ddagger\, y$ for $N = 25$



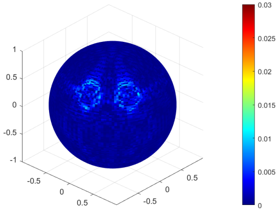
(c) $|x - R^\ddagger\, y|$ for $N = 25$



(d) Data $y = R\,x$

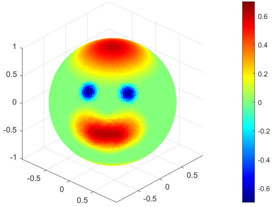


(e) $R^\ddagger\, y$ for $N = 40$



(f) $|x - R^\ddagger\, y|$ for $N = 40$

Reconstruction Evaluation, Noise Level 20%



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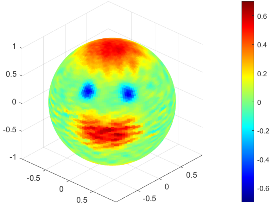
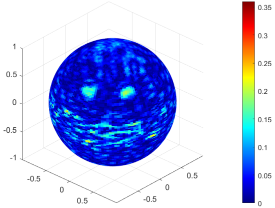
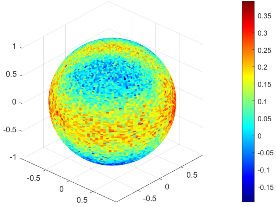


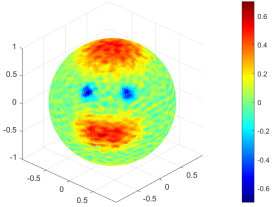
(b) $R^{\ddagger} U_{\alpha}\, y^{\delta}$ for $N = 25,\ \alpha = 0.064$



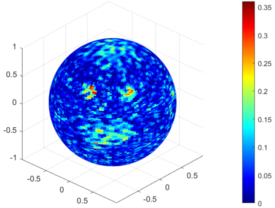
(c) $|x - R^{\ddagger} U_{\alpha}\, y^{\delta}|$ for $N = 25$



(d) Noisy data $y^{\delta} = R\,x + \delta$



(e) $R^{\ddagger} U_{\alpha}\, y^{\delta}$ for $N = 40,\ \alpha = 0.14$



(f) $|x - R^{\ddagger} U_{\alpha}\, y^{\delta}|$ for $N = 40$

Conclusions

- We derived a novel frame decomposition of the Funk-Radon transform utilizing trigonometric basis functions $b_{n,k}$ on the unit sphere and suitable embedding operators in Sobolev spaces.
- This decomposition does not involve the spherical harmonics and leads to an explicit inversion formula for the Funk-Radon transform.
- Our numerical examples show promising reconstruction results even in the case of very large noise by including regularization.

Future research

- Apply other forms of regularization (e.g. filter applied to the frame coefficients) to avoid the computationally expensive spherical harmonics entirely
- Investigate the possibility of better localized frames

Thank you for your attention!

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